SECTION 5.1: AREA

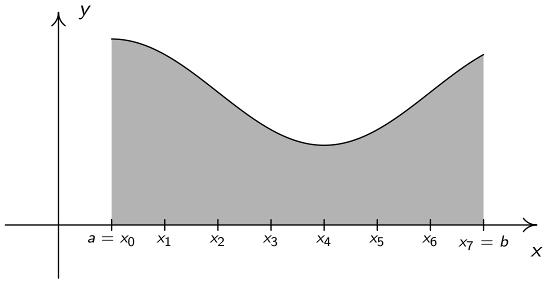
Name:____

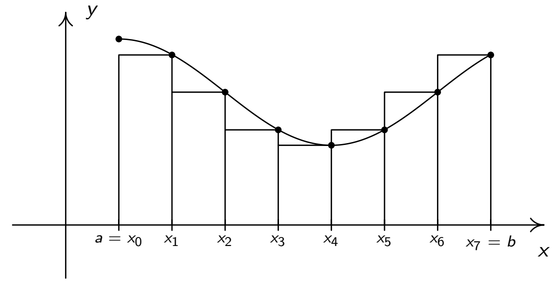
The Area Problem: We begin our journey to definite integrals by wondering how to find the area between the graph of a continuous function $y = f(x)$ and the x-axis over some closed, finite interval $[a, b]$. Since we don't know any formulas for 'fancy' regions, we stick to what we know - rectangles.

To keep things simple, we divide $[a, b]$ into n equal pieces (subintervals), and use the right-endpoints of each piece to determine the height of the rectangles.

We let x_i represent the right endpoint of the ith subinterval, so the height of the ith rectangle is $f(x_i)$. The width of the ith rectangle is the length of the ith subinterval and is denoted $\Delta x_i = x_i - x_{i-1}$.

Below is a depiction of RS_7, a 'right endpoint sum' using 7 (equally spaced) subintervals.[1]



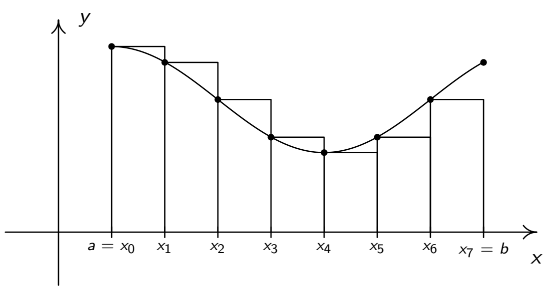


Visualizing RS_7, a 'right endpoint sum.'

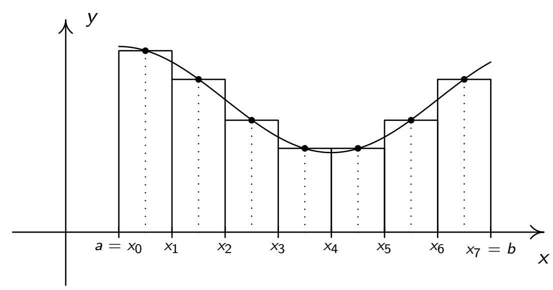
In symbols:

$$\text{Area} \approx f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + \ldots + f(x_7)\Delta x_7 = \sum_{i=1}^{7} f(x_i)\Delta x_i$$

Instead of using the right endpoint of the subinterval as the location to sample the function, we could also use the left endpoint, or even the midpoint, of the interval, as indicated below:



Visualizing LS_7, a 'left endpoint sum.'



Visualizing MS_7, a 'midpoint sum.'

No matter what values we use, we are approximating the area under the curve by the sum of the areas of the rectangles. To get a better approximation of the actual area, we can use more rectangles.

STRATEGY: Use pre-calculus to obtain formulas for RS_n, LS_n and/or MS_n and then take the limit as $n \to \infty$.

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[1] On intervals over which the function is **increasing**, the area of rectangles **overestimates** the area we want and is called an **upper sum** on that interval. Likewise, on intervals over which the function is **decreasing**, we find the area of the rectangles **underestimates** the area we want and is called a **lower sum** on that interval.

We summarize some of the important notations below:

Summary of Formulas for Endpoint and Midpoint Sums, RS_n, LS_n, and MS_n.

- n - number of rectangles
- A **partition** of $[a, b]$ is a set of points $\Delta = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ with

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

  If the x_i are equally spaced, the partition is called a **regular** partition.

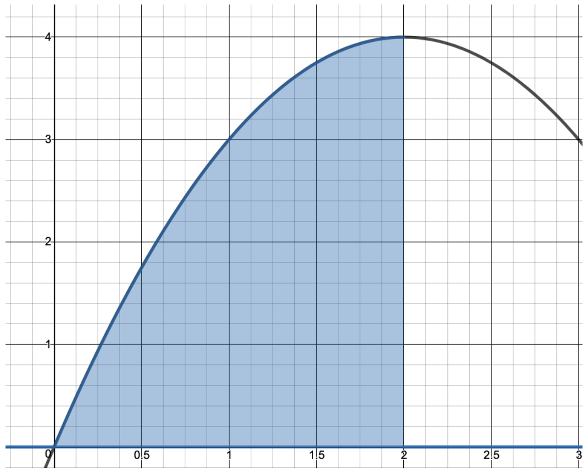
- Given a partition Δ, we set $\Delta x_i = x_i - x_{i-1}$. Geometrically, Δx_i is width of ith rectangle.
  For a regular partition, $\Delta x_i = \dfrac{b-a}{n}$.
  The **norm** of a partition, $\|\Delta\|$ is the largest of the Δx_i.
- x_i^* - some value in the i th subinterval. Some popular options are:
  - right endpoints: $x_i^* = x_i = a + i\,\Delta x_i$
  - left endpoints: $x_i^* = x_{i-1} = a + (i-1)\,\Delta x_i$
  - midpoints: $x_i^* = \frac{1}{2}\,(x_i + x_{i-1}) = a + \left(i - \frac{1}{2}\right)\Delta x_i$
- $f(x_i^*)$ - height of ith rectangle
- Area $\approx$ the sum of the area of the rectangles $= \displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x_i$

Next, we summarize some important properties of summation below:

Properties of (finite) Sums

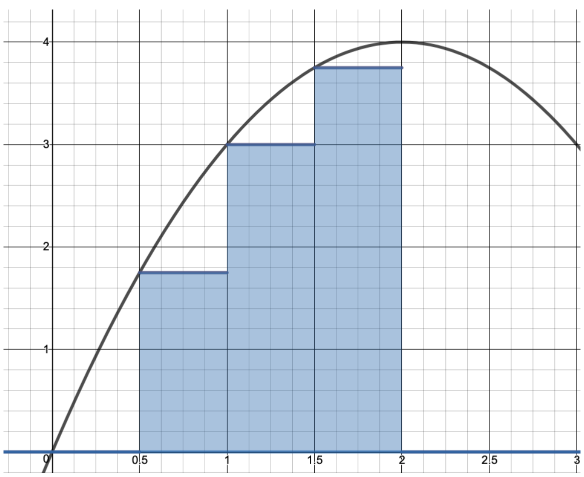
- $\displaystyle\sum_{i=1}^{n} c\,A_i = c\sum_{i=1}^{n} A_i$
- $\displaystyle\sum_{i=1}^{n} A_i \pm B_i = \sum_{i=1}^{n} A_i \pm \sum_{i=1}^{n} B_i$
- $\displaystyle\sum_{i=1}^{n} c = cn$
- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

EXAMPLE 1: Let A be the area between the graph of $f(x) = 4x - x^2$ and the x-axis over the interval $[0, 2]$:

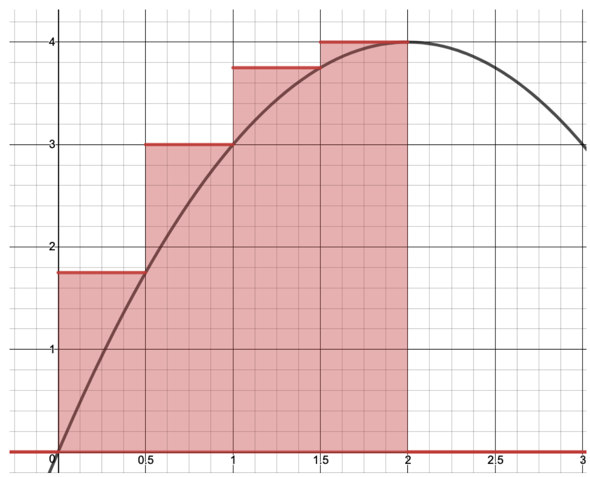


Sketch of the area A.

Sketch and compute LS_4 and RS_4.



LS_4

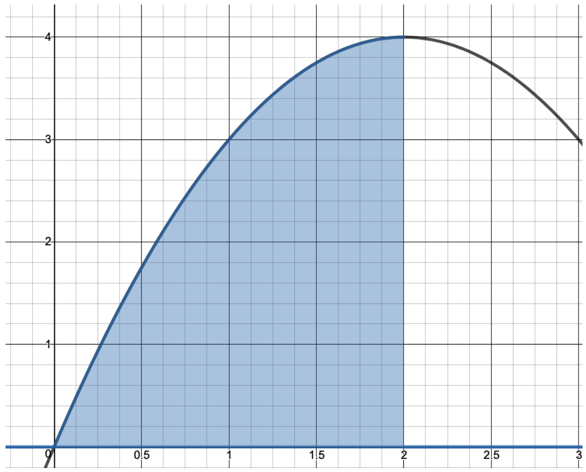


RS_4

$LS_4 = f(0)\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\frac{1}{2} + f(1)\frac{1}{2} + f\left(\frac{3}{2}\right)\frac{1}{2} = (0)\left(\frac{1}{2}\right) + \left(\frac{7}{4}\right)\frac{1}{2} + (3)\frac{1}{2} + \left(\frac{15}{4}\right)\frac{1}{2} = \frac{17}{4} = 4.25 \text{ units}^2.$

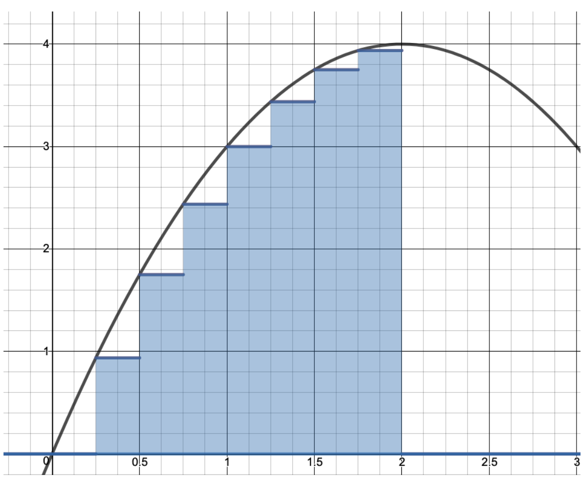
$RS_4 = f\left(\frac{1}{2}\right)\frac{1}{2} + f(1)\frac{1}{2} + f\left(\frac{3}{2}\right)\frac{1}{2} + f(2)\frac{1}{2} = \left(\frac{7}{4}\right)\frac{1}{2} + (3)\frac{1}{2} + \left(\frac{15}{4}\right)\frac{1}{2} + (4)\frac{1}{2} = \frac{25}{4} = 6.25 \text{ units}^2.$

EXAMPLE 2: Let A be the area between the graph of $f(x) = 4x - x^2$ and the x-axis over the interval $[0, 2]$:



Sketch of the area A.

Sketch and compute LS_8 and RS_8.



LS_8

RS_8

$LS_8 = f(0)\left(\frac{1}{4}\right) + f\left(\frac{1}{4}\right)\frac{1}{4} + f\left(\frac{1}{2}\right)\frac{1}{4} + f\left(\frac{3}{4}\right)\frac{1}{4} + f(1)\frac{1}{4} + f\left(\frac{5}{4}\right)\frac{1}{4} + f\left(\frac{3}{2}\right)\frac{1}{4} + f\left(\frac{7}{4}\right)\frac{1}{4} = \ldots = \frac{77}{16} = 4.8125$ units^2.

$RS_8 = f\left(\frac{1}{4}\right)\frac{1}{4} + f\left(\frac{1}{2}\right)\frac{1}{4} + f\left(\frac{3}{4}\right)\frac{1}{4} + f(1)\frac{1}{4} + f\left(\frac{5}{4}\right)\frac{1}{4} + f\left(\frac{3}{2}\right)\frac{1}{4} + f\left(\frac{7}{4}\right)\frac{1}{4} + f(2)\left(\frac{1}{4}\right) = \ldots = \frac{93}{16} = 5.8125$ units^2.

EXAMPLE 3: Let A be the area between the graph of $f(x) = 4x - x^2$ and the x-axis over the interval $[0, 2]$:

1. Find and simplify an expression for Δx_i: $\Delta x_i = \dfrac{b-a}{n} = \dfrac{2-0}{n} = \dfrac{2}{n}$

2. Find and simplify an expression for x_{i-1}: $x_{i-1} = a + (i-1)\,\Delta x_i = (i-1)\dfrac{2}{n} = \dfrac{2i}{n} - \dfrac{2}{n}$

3. Find and simplify an expression for $f(x_{i-1})$:

$$f(x_{i-1}) = 4x_{i-1} - (x_{i-1})^2 = 4\left(\frac{2i}{n} - \frac{2}{n}\right) - \left(\frac{2i}{n} - \frac{2}{n}\right)^2 = \ldots = \frac{8i}{n} - \frac{8}{n} - \frac{4i^2}{n^2} + \frac{8i}{n^2} - \frac{4}{n^2}$$

4. Find an expression for $LS_n = \displaystyle\sum_{i=1}^{n} f(x_{i-1})\Delta x_i = \sum_{i=1}^{n}\left(\frac{8i}{n} - \frac{8}{n} - \frac{4i^2}{n^2} + \frac{8i}{n^2} - \frac{4}{n^2}\right)\frac{2}{n}$

5. Use properties of sums to simplify LS_n:

$$\begin{aligned}
LS_n &= \sum_{i=1}^{n}\left(\frac{8i}{n} - \frac{8}{n} - \frac{4i^2}{n^2} + \frac{8i}{n^2} - \frac{4}{n^2}\right)\frac{2}{n} \\
&= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{8i}{n} - \frac{8}{n} - \frac{4i^2}{n^2} + \frac{8i}{n^2} - \frac{4}{n^2}\right) \\
&= \frac{2}{n}\left[\sum_{i=1}^{n}\frac{8i}{n} - \sum_{i=1}^{n}\frac{8}{n} - \sum_{i=1}^{n}\frac{4i^2}{n^2} + \sum_{i=1}^{n}\frac{8i}{n^2} - \sum_{i=1}^{n}\frac{4}{n^2}\right] \\
&= \frac{2}{n}\left[\frac{8}{n}\sum_{i=1}^{n} i - \frac{8}{n}\sum_{i=1}^{n} 1 - \frac{4}{n^2}\sum_{i=1}^{n} i^2 + \frac{8}{n^2}\sum_{i=1}^{n} i - \frac{4}{n^2}\sum_{i=1}^{n} 1\right] \\
&= \frac{2}{n}\left[\frac{8}{n}\left(\frac{n(n+1)}{2}\right) - \frac{8}{n}(n) - \frac{4}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{4}{n^2}(n)\right] \\
&= \frac{2}{n}\left[4(n+1) - 8 - \frac{2(n+1)(2n+1)}{3n} + \frac{4(n+1)}{n} - \frac{4}{n}\right] \\
LS_n &= \frac{8(n+1)}{n} - \frac{16}{n} - \frac{4(n+1)(2n+1)}{3n^2} + \frac{8(n+1)}{n^2} - \frac{8}{n^2}
\end{aligned}$$

6. Find the area A by finding $\displaystyle\lim_{n\to\infty} LS_n$.

$$\lim_{n\to\infty} LS_n = \lim_{n\to\infty}\left[\frac{8(n+1)}{n} - \frac{16}{n} - \frac{4(n+1)(2n+1)}{3n^2} + \frac{8(n+1)}{n^2} - \frac{8}{n^2}\right] = \ldots = 8 - \frac{8}{3} = \frac{16}{3} = 5.\overline{3}\text{ units}^2.$$

EXAMPLE 4: Let A be the area between the graph of $f(x) = 4x - x^2$ and the x-axis over the interval $[0, 2]$:

1. Find and simplify an expression for Δx_i: $\Delta x_i = \dfrac{b-a}{n} = \dfrac{2-0}{n} = \dfrac{2}{n}$

2. Find and simplify an expression for x_i: $x_i = a + i\,\Delta x_i = \dfrac{2i}{n}$

3. Find and simplify an expression for $f(x_i)$:

$$f(x_i) = 4x_i - (x_i)^2 = 4\left(\frac{2i}{n}\right) - \left(\frac{2i}{n}\right)^2 = \ldots = \frac{8i}{n} - \frac{4i^2}{n^2}$$

4. Find an expression for $RS_n = \displaystyle\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n}\left(\frac{8i}{n} - \frac{4i^2}{n^2}\right)\frac{2}{n}$

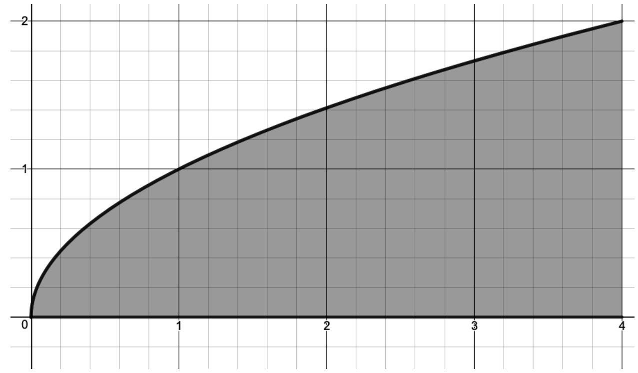
5. Use properties of sums to simplify RS_n:

$$\begin{aligned}
RS_n &= \sum_{i=1}^{n}\left(\frac{8i}{n} - \frac{4i^2}{n^2}\right)\frac{2}{n} \\
&= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{8i}{n} - \frac{4i^2}{n^2}\right) \\
&= \frac{2}{n}\left[\sum_{i=1}^{n}\frac{8i}{n} - \sum_{i=1}^{n}\frac{4i^2}{n^2}\right] \\
&= \frac{2}{n}\left[\frac{8}{n}\sum_{i=1}^{n} i - \frac{4}{n^2}\sum_{i=1}^{n} i^2\right] \\
&= \frac{2}{n}\left[\frac{8}{n}\left(\frac{n(n+1)}{2}\right) - \frac{4}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right] \\
&= \frac{2}{n}\left[4(n+1) - \frac{2(n+1)(2n+1)}{3n}\right] \\
RS_n &= \frac{8(n+1)}{n} - \frac{4(n+1)(2n+1)}{3n^2}
\end{aligned}$$

6. Find the area A by finding $\displaystyle\lim_{n\to\infty} RS_n$.

$$\lim_{n\to\infty} RS_n = \lim_{n\to\infty}\left(\frac{8(n+1)}{n} - \frac{4(n+1)(2n+1)}{3n^2}\right) = \ldots = 8 - \frac{8}{3} = \frac{16}{3} = 5.\overline{3}\text{ units}^2.$$

EXAMPLE 5: Consider the area between $y = f(x) = \sqrt{x}$ and the x-axis over the interval $[0, 4]$.



Dividing the interval up into equal pieces causes problems, since $f(x_i) = \sqrt{x_i}$ would result in terms involving square roots, and we have no formulas for these sums.

One work-around is to use right endpoints which are always perfect squares:

$$x_i = \frac{4i^2}{n^2}, \quad i = 0, 1, \ldots n$$

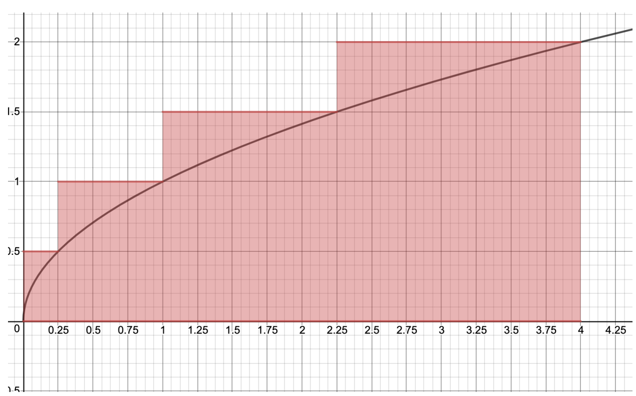
While this simplifies the height of the rectangles,

$$f(x_i) = \sqrt{\frac{4i^2}{n^2}} = \frac{2i}{n},$$

it leads to complications with the width of the rectangles

$$\Delta x_i = x_i - x_{i-1} = \frac{4i^2}{n^2} - \frac{4(i-1)^2}{n^2} = \frac{8i-4}{n^2}$$

Use the partition: $x_i = \dfrac{4i^2}{16}$, $i = 0, 1, \ldots 4$ to sketch and find RS_4:



RS_4

$$RS_4 = f\left(\tfrac{1}{4}\right)\tfrac{1}{4} + f(1)\tfrac{3}{4} + f\left(\tfrac{9}{4}\right)\tfrac{5}{4} + f(4)\tfrac{7}{4} = \left(\tfrac{1}{2}\right)\tfrac{1}{4} + (1)\tfrac{3}{4} + \left(\tfrac{3}{2}\right)\tfrac{5}{4} + (2)\tfrac{7}{4} = \frac{25}{4} = 6.25 \text{ units}^2$$

NOTE: Since we don't have a regular partition in this case, we note $\|\Delta\| =$ the largest $\Delta x_i = \Delta x_4 = \frac{7}{4}$.

EXAMPLE 6: (VIDEO) Consider the area between $y = f(x) = \sqrt{x}$ and the x-axis over the interval $[0, 4]$.

Using the partition: $x_i = \dfrac{4i^2}{n^2}$ for $i = 0, 1, \ldots n$, we get:

$$f(x_i) = \sqrt{\frac{4i^2}{n^2}} = \frac{2i}{n} \qquad \text{and} \qquad \Delta x_i = x_i - x_{i-1} = \frac{8i-4}{n^2}$$

1. Find and simplify a formula for RS_n:

$$\begin{aligned}
RS_n &= \sum_{i=1}^{n} f(x_i)\Delta x_i \\
&= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{8i-4}{n^2}\right) \\
&= \frac{1}{n^3}\sum_{i=1}^{n} 2i(8i-4) \\
&= \frac{1}{n^3}\sum_{i=1}^{n} \left(16i^2 - 8i\right) \\
&= \frac{1}{n^3}\left[\sum_{i=1}^{n} 16i^2 - \sum_{i=1}^{n} 8i\right] \\
&= \frac{1}{n^3}\left[16\sum_{i=1}^{n} i^2 - 8\sum_{i=1}^{n} i\right] \\
&= \frac{1}{n^3}\left[16\left(\frac{n(n+1)(2n+1)}{6}\right) - 8\left(\frac{n(n+1)}{2}\right)\right] \\
&= \frac{1}{n^3}\left[\frac{8n(n+1)(2n+1)}{3} - 4n(n+1)\right] \\
RS_n &= \frac{8(n+1)(2n+1)}{3n^2} - \frac{4(n+1)}{n^2}
\end{aligned}$$

2. Find the area A by finding $\lim\limits_{n\to\infty} RS_n$.

$$\lim_{n\to\infty} RS_n = \lim_{n\to\infty}\left(\frac{8(n+1)(2n+1)}{3n^2} - \frac{4(n+1)}{n^2}\right) = \ldots = \frac{16}{3} - 0 = \frac{16}{3} = 5.\overline{3} \text{ units}^2.$$

HOMEWORK:(VIDEO) Find the area beneath the curve $y = f(x)$ on the interval $[a, b]$ using the limit process:

1. $f(x) = 4 - x$ over the interval $[0, 4]$.

   Ans: 8 units^2

2. $f(x) = 3x^2$ over the interval $[1, 3]$.

   Ans: 26 units^2

3. $f(x) = 12 - x - x^2$ over the interval $[0, 3]$.

   Ans: 22.5 units^2

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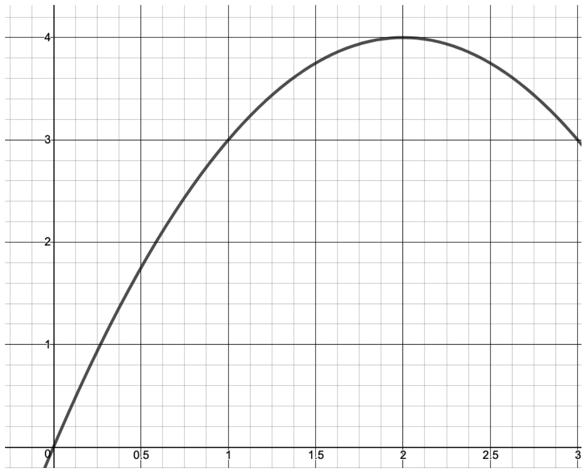
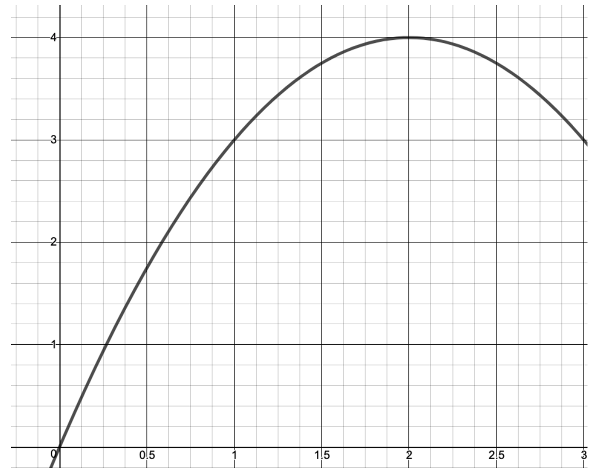
Sketch and compute LS_4 and RS_4.

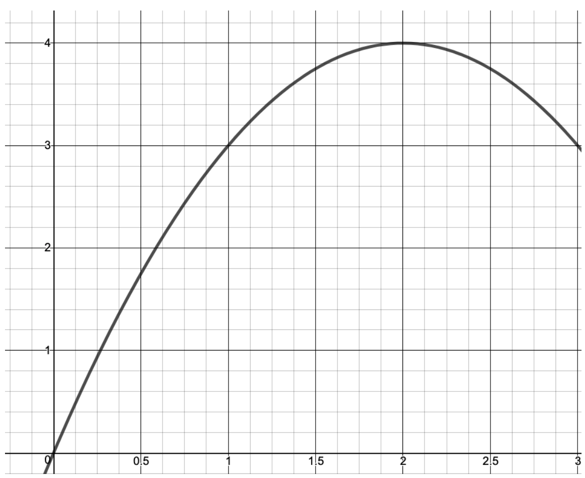


LS_4

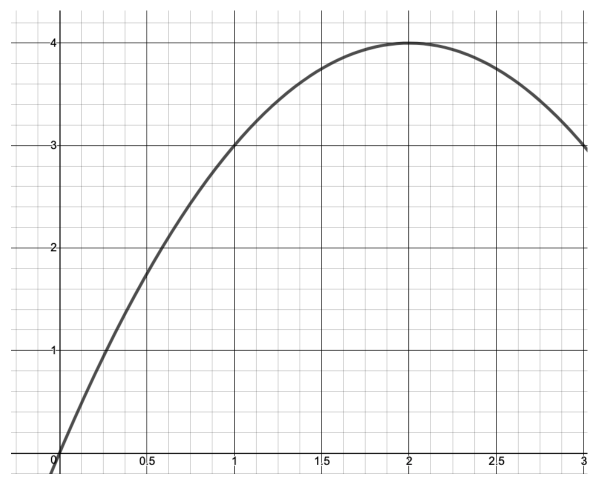


RS_4

Sketch and compute LS_8 and RS_8.



LS_8



RS_8

Use the partition: $x_i = \dfrac{4i^2}{16}$, $i = 0, 1, \ldots 4$ to sketch and find RS_4: